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Crumpled and flat regimes in a random surface model

G Gonnella and A Maritan†

Dipartimento di Fisica, Università di Bari, Italy and Istituto Nazionale di Fisica Nucleare Sezione di Bari, Italy

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Abstract. A spherical like model of a D-dimensional random surface embedded in d-dimensional Euclidean space is studied in detail. The embedding weight depends on an attractive term between the nearest neighbours and on a repulsive one between some of the next to the nearest neighbours of the network. The repulsive term mimics an extrinsic curvature energy for surface configurations. Crumpled and flat regimes are found, and, if $D \leqslant 2$, only the former survives in the thermodynamic limit. The model can be seen as the $d \to \infty$ limit of a more realistic model where the $1/d$ corrections stabilize the flat regime in the thermodynamic limit at least for $D = 2$.

1. Introduction

Crystalline membranes are D-dimensional networks of particles with fixed connectivity embedded into the d-dimensional Euclidean space. They can be regarded as the simplest generalization of linear polymers [1]. The recent interest [2] in the study of crystalline membranes is due to the fact that, unlike linear polymers or fluid membranes, their elastic forces together with the bending energy can stabilize a low temperature flat phase [3]. At sufficiently high temperature there is a 'crumpling transition' [4, 5] into a disordered phase with infinite Hausdorff dimension (crumpled phase). Numerical simulations [5] indicate that such a phase transition occurs even for two-dimensional membranes at variance with standard two-dimensional systems with a continuous symmetry group (in the present case the rotation group of the embedding space). In other words the famous Mermin–Wagner [6] theorem does not hold in the case of crystalline membranes.

The constraint of fixed connectivity is realized by assigning a d-dimensional coordinate $\boldsymbol{X}_i$ to each node i of a D-dimensional network, and a bond energy $V(|\boldsymbol{X}_i - \boldsymbol{X}_j|)$, diverging as $|\boldsymbol{X}_i - \boldsymbol{X}_j|$ goes to infinity. The total energy also includes a bending elastic term [7] whose strength is given by the rigidity constant k. The behaviour of k under renormalization due to thermal fluctuations is crucial for the existence of the flat phase. If for example $V(r) = 0$ and the energy depends on the surface area (fluid membranes), then the rigidity k, in a model where the network is approximated by a continuous manifold, has been shown [4] to decrease on large scales and the flat phase does not exist in $D = 2$. The situation is different in crystalline membranes where an effective

† Mail address: Dipartimento Fisica, Via F Marzolo 8, 35131 Padova, Italy.

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long range interaction is generated, leading to a renormalized rigidity which grows up on large scales, so that the crumpling transition is made possible [3].

Bi-dimensional membranes with solid internal order embedded in three-dimensional Euclidean space have been extensively studied by Monte Carlo simulations in [5]. The energy configuration was taken as

$$E=-k\sum_{\langle\alpha,\beta\rangle}(\boldsymbol{n}_\alpha\cdot\boldsymbol{n}_\beta-1)+\sum_{\langle i,j\rangle}V(|\boldsymbol{X}_i-\boldsymbol{X}_j|)$$

where the nodes i and j belong to a triangular network, $\boldsymbol{n}_\alpha$ is the unit vector normal to the αth triangle and V is a potential between the nearest-neighbouring nodes. There is a rather clear evidence [5] for a crumpling transition at a given k_c between a 'low temperature' ($k>k_c$) flat phase when the membrane radius of gyration R grows linearly with the linear size L of the network and a 'high temperature' crumpled phase when $R^2\sim\ln L$. In the model considered in [5] $V(r)=0$ when $1<r<\sqrt{3}$ and $V(r)=\infty$ otherwise (tethered membranes). However it is believed that the crumpling transition exists for a wide class of potentials.

The aim of this paper is to investigate the existence of the crumpling transition in an explicitly solvable lattice model with a repulsive potential between the next to nearest neighbours substituting for the interaction between the normals of triangles. Such a choice appeared in the numerical simulations of [8].

In section 2 the model is defined and the linear chain case is solved. The bi-dimensional problem is treated in section 3, where the existence of a crumpled 'regime' and of a flat 'regime' will be found maintaining the size of the membrane as finite. The model proposed is analogous to the spherical model and, as such, it does not exhibit a phase transition in $D=2$ when the size of the membrane becomes infinite. (A fluid membrane version of the spherical model, treating the surface as a continuum in the Monge representation, has been already discussed some time ago in [9]). A true phase transition exists for $D=3$, and the corresponding critical behaviour will be discussed in section 4. In the last section the results will be commented and it will be observed that our model can be viewed as the $d\to\infty$ limit of a more physical model, just like the spherical model is equivalent to the O(N) model in the large N limit [10]. Corrections of order $1/d$ can also be calculated by a straightforward generalization of the standard $1/N$ expansion. Finally the appendices contain technical details.

2. The definition of the model and the $D=1$ case

The statistical mechanics of the embeddings of a D-dimensional network into R^d can be described by the Boltzmann weight e^{-H}/Z with

$$H=\frac{a^2}{N}\left[\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right]^2+b\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2-k\sum_{\ll i,i\gg}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2 \qquad (2.1)$$

where i and j are nodes of the network and $\boldsymbol{X}_i\in\mathbf{R}^d$. The symbols $\langle\ldots\rangle$ and $\ll\ldots\gg$ are respectively used to indicate the nearest neighbours and some of the next neighbours of the network, k is the rigidity constant and N is the total number of sites. The partition function Z includes a δ function which fixes the centre of mass of the network and eliminates the infinite d-dimensional volume coming out from integration, due to the translational invariance of (1.1).

In the one-dimensional case the partition function is

$$Z=\int\prod_{i=0}^{N-1}\mathrm{d}^dX_i\,\delta\left(\frac{1}{N}\sum_{i=0}^{N-1}X_i\right)$$
$$\times\exp\left\{-\frac{a^2}{N}\left[\sum_{i=0}^{N-1}(X_i-X_{i+1})^2\right]^2-\frac{b}{2}\sum_{i=0}^{N-1}(X_i-X_{i+1})^2\right.$$
$$\left.+\frac{k}{2}\sum_{i=0}^{N-1}(X_i-X_{i+2})^2-\frac{\beta}{2N}\sum_{i,j=0}^{N-1}(X_i-X_j)^2\right\} \tag{2.2}$$

where periodic boundary conditions ($X_{i+N}=X_i$) will be considered. The term proportional to β is a source term for the radius of gyration (see e.g. (2.9)). The model can be seen as a realization of the spherical model, and, by the same arguments used by Stanley [10], it can be shown equivalent to a model with the quartic interaction $a^2/d\ \Sigma_{\langle i,j\rangle}|x_i-x_j|^4$ in the limit $d\to\infty$ (see section 5). The quartic term of the energy (2.1) can be interpreted as the compression energy due to ripples on the network (see [9] where the effect of such a term on the behaviour of the surface tension in fluid membranes has been studied).

By applying the Hubbard–Stratonivich transformation to the quartic term in (2.2) the partition function can be rewritten in the following way

$$Z=\frac{1}{2\mathrm{i}}\sqrt{\frac{N}{\pi}}\int\prod_i\mathrm{d}^d X_i\delta\left(\frac{1}{N}\sum_i X_i\right)\int_{-\mathrm{i}\infty+c}^{\mathrm{i}\infty+c}\mathrm{d}z$$
$$\times\exp\left(\frac{z^2N}{4}-\left(\frac{b}{2}+za\right)\sum_i(X_i-X_{i+1})^2\right.$$
$$\left.+\frac{k}{2}\sum_i(X_i-X_{i+2})^2-\frac{\beta}{2N}\sum_{i,j}(X_i-X_j)^2\right)$$
$$=\frac{N^{(d+1)/2}}{\mathrm{i}\sqrt{4\pi}}(2\pi)^{d(N-1)/2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}z\,\exp[-Nf(z)] \tag{2.3}$$

where

$$f(z)=-\frac{z^2}{4}+\frac{d}{2N}\sum_{m=1}^{N-1}\ln D(m;z,\beta) \tag{2.4}$$

and

$$D(m;z,\beta)=2(b+2za)\left(1-\cos\frac{2\pi m}{N}\right)-2k\left(1-\cos\frac{4\pi m}{N}\right)+2\beta. \tag{2.5}$$

In the last step of (2.3) the Fourier series of X_i has been introduced before performing the integration over the 'normal modes'.

In (2.3) c is arbitrary as far as $D(m;\mathrm{Re}\,z;0)\equiv D(m;\mathrm{Re}\,z)$ is greater than zero for $m=1,\ldots,N-1$, which is true if $D(1;\mathrm{Re}\,z)>0$. When $N\gg1$, Z can be evaluated by applying standard saddle point methods to (2.3). The saddle point equation $(\mathrm{d}f(z)/\mathrm{d}z)_{z=\bar{z}}=0$ gives (in the following $b=1$)

$$\bar{z}=\frac{2da}{N}\sum_{m=1}^{N-1}\left[1+2\bar{z}a-4k+2k\left(1-\cos\frac{2\pi m}{N}\right)\right]^{-1} \tag{2.6}$$

which, in terms of $x=(1+2\bar{z}a-4k)/2k$, becomes

$$\frac{k\bar{z}}{ad}=\frac{k}{2da^2}(2kx-1+4k)$$

$$=\frac{1}{N}\sum_{m=1}^{N-1}\left(1+x-\cos\frac{2\pi m}{N}\right)^{-1}\equiv G(x). \tag{2.7}$$

The condition $D(1;z)>0$ leads to

$$x+1-\cos\frac{2\pi}{N}>0. \tag{2.8}$$

The spatial configuration of the network will be described by the ratio R^2/l^2, where R is the mean squared radius of gyration

$$R^2=\left\langle\frac{1}{N^2}\sum_{i,j}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right\rangle \tag{2.9}$$

and l is the link mean length

$$l^2=\left\langle\frac{1}{n_{\mathrm{L}}}\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right\rangle \tag{2.10}$$

where n_{L} is the number of the links of the network (which is N in the present case). Different 'phases' or regimes will be characterized by different behaviours of R^2/l^2 as functions of N.

From (2.2)-(2.5) and from the definitions (2.9) and (2.10) one gets

$$R^2=-\frac{2}{N}\frac{\mathrm{d}}{\mathrm{d}\beta}\ln Z\bigg|_{\beta=0}=2\frac{\partial f(\bar{z})}{\partial\beta}\bigg|_{\beta=0}$$

$$=\frac{d}{2kN}\sum_{m=1}^{N-1}\left(1-\cos\frac{2\pi m}{N}\right)^{-1}\left(1+x-\cos\frac{2\pi m}{N}\right)^{-1}\bigg|_{z=\bar{z}} \tag{2.11}$$

and

$$l^2=-\frac{2}{n_{\mathrm{L}}}\frac{\mathrm{d}}{\mathrm{d}b}\ln Z=\frac{1}{a}\frac{\partial}{\partial z}\left(f(z)+\frac{z^2}{4}\right)\bigg|_{z=\bar{z}}=\frac{\bar{z}}{2a} \tag{2.12}$$

where the saddle point equation has also been used. In calculating $G(x)$ and R^2 the two cases $x\ll 1$ (corresponding to $\bar{z}\simeq(4k-1)/2a$, which is the minimum value from the condition (2.8) when $N\to\infty$) and $x\simeq 1$ will be separately considered. If $x\ll 1$, $G(x)$ can be approximated by

$$G(x)\simeq\frac{2}{N}\sum_{m=1}^{(N-1)/2}\frac{1}{x+2\pi^2m^2/N^2}\simeq N\sum_{m=1}^{\infty}\frac{1}{t^2+(m\pi)^2} \tag{2.13}$$

with $t=N\sqrt{x/2}$. The application of the theorem of residues gives

$$G(x)=N(t\coth t-1)/2t^2 \tag{2.14}$$

so that

$$G(x) \simeq \begin{cases} 1/\sqrt{2x} & \text{if } N^2x \gg 1 (t \to \infty) \qquad (2.15a) \\ N/6 & \text{if } N^2x \ll 1 (t \to 0). \qquad (2.15b) \end{cases}$$

The saddle point equation (2.7) can now be solved giving estimates of the scales of k corresponding to various regimes. By inserting (2.15a) into (2.7) one obtains

$$x = \frac{2d^2a^4}{k^2(4k-1)^2} \tag{2.16}$$

with the condition $N^{-2} \ll x \ll 1$ equivalent to $1 \ll k \ll \sqrt{N}$. The scale $k \simeq \sqrt{N}$ corresponds to (2.15b), as it can be easily seen from (2.7). The computation of R^2 is similar. Using $k \ll \sqrt{N}$

$$\begin{aligned} R^2 &\simeq \frac{dN^3}{4k} \sum_{m=1}^{\infty} \frac{1}{(\pi m)^2(t^2+(\pi m)^2)} \\ &= \frac{dN^3}{24kt^4}(t^2 - 3t \coth t + 3) \simeq \frac{2Nk^3}{3da^4} \end{aligned} \tag{2.17}$$

where (2.16) with $k \gg 1$ has also been used. Finally, from (2.12) and (2.17)

$$\frac{R^2}{l^2} \simeq \frac{2k^2}{3da^2} N \qquad 1 \ll k \ll \sqrt{N}. \tag{2.18}$$

A different situation occurs for $x < 0$ and $|x| \ll 1$. From (2.7)

$$G(x) \simeq N \sum_{m=1}^{\infty} \frac{1}{(m\pi)^2 - t^2} = \frac{N}{2} \sum_{|m| \geq 2} \frac{1}{(m\pi)^2 - t^2} + \frac{N}{\pi^2 - t^2} \tag{2.19}$$

where, from (2.8), $t = N\sqrt{|x|/2} < \pi$. In the limit $t \to \pi^-$

$$G(x) \simeq \frac{N}{2\pi} \frac{1}{\pi - t} \tag{2.20}$$

which, inserted into (2.7), gives $k \gg \sqrt{N}$. The mean squared radius of gyration, R^2, can be analogously calculated and

$$\frac{R^2}{l^2} \simeq \frac{N^2}{2\pi^2} \qquad k \gg \sqrt{N} \tag{2.21}$$

corresponding to the 'flat regime'. Equation (2.18) corresponds to intermediate spatial configurations, while the totally crumpled regime comes out for $x \gtrsim 1$, when

$$R^2 \simeq \frac{d}{kNx} \sum_{m=1}^{(N-1)/2} \frac{1}{1 - \cos(2\pi m/N)} \simeq \frac{dN}{12kx}. \tag{2.22}$$

The corresponding saddle point equation gives $k\bar{z}/ad \simeq 1/x$, so that $l^2 = d/2kx$, and

$$\frac{R^2}{l^2} \simeq \frac{N}{6} \qquad k \ll 1. \tag{2.23}$$

The above results imply that at a given value of the rigidity $k \gg 1$, the linear polymer is stretched on length scale of order k^2. This is also in accord with the general fact that the tangent vectors of a polymer develop long-range correlations only when the curvature parameter tends to infinity [11] (for a new class of random walks with curvature dependent action see [12]). Since $l^2=\bar{z}/2a$, at the flattening scale $k\sim\sqrt{N}$ the chain is near to an instability because $\bar{z}\sim k$ (see (2.8); the same phenomenon occurs also at $D=2$). The phenomenon can be related to a lack of scale invariance in the curvature term of (2.2). When the curvature energy is scale invariant the length of subsequent steps are uncorrelated and the mean interatomic distance l is independent of the curvature [12]. However, this discussion does not afflict the model of this paper, which has been introduced mainly to study the dependence of the persistence length on the rigidity k.

3. The random surface model

3.1. Radius of gyration

Let a $D=2$ network consist of equilateral triangles joined at their edges, with the topology of a torus. On each side of the network (the opposite sides are identified) there will be $L+1$ vertices; vertices are specified by two integer coordinates (n_1, n_2) with $0\leqslant n_1,\ n_2\leqslant L$. In the following L will be taken odd for simplicity. The couple of integers (n_1, n_2) will be abbreviated by a single index $i, j, \ldots$, when no ambiguity arises (see figure 1). The symbol $\langle\langle i,j\rangle\rangle$ will be used when i and j belong to two different triangles with a common side. The partition function reads as

$$Z=\int\prod_i \mathrm{d}^d X_i\delta\left(\frac{1}{L^2}\sum_i \boldsymbol{X}_i\right)$$

$$\times\exp\left[-\frac{a^2}{L^2}\left(\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right)^2-b\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2+k\sum_{\langle\langle i,j\rangle\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right]$$

$$=\frac{L}{2\mathrm{i}\sqrt{\pi}}\int\prod_i \mathrm{d}^d X_i\delta\left(\frac{1}{L^2}\sum_i \boldsymbol{X}_i\right)$$

$$\int_{-\mathrm{i}\infty+c}^{\mathrm{i}\infty+c}\mathrm{d}z\,\exp\left(\frac{z^2L^2}{4}-(b+za)\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2+k\sum_{\langle\langle i,j\rangle\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right) \qquad (3.1)$$

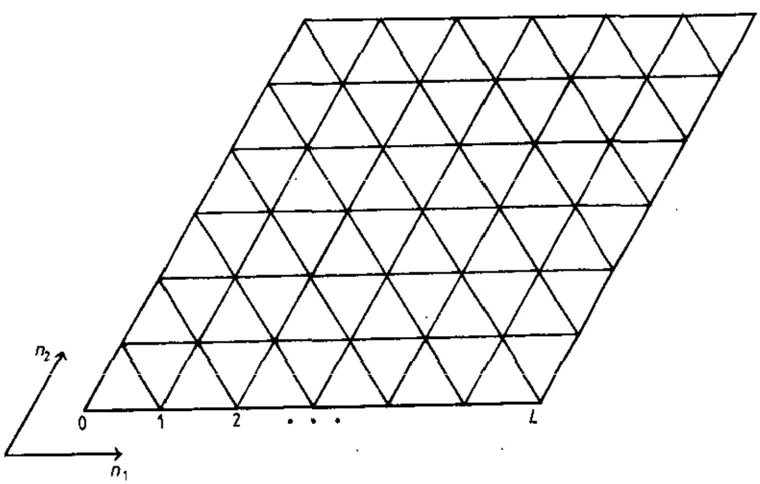

Figure 1.

with c arbitrary. The order of the integrations can be interchanged only if

$$[b+a\ \mathrm{Re}\ z]\sum_{\langle i,j\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2-k\sum_{\langle\langle i,j\rangle\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\equiv\sum_{i,j}\boldsymbol{X}_iD_{ij}\boldsymbol{X}_j \tag{3.2}$$

is strictly positive. The matrix (D_{ij})

$$D_{ij}=\begin{cases}6(b+a\ \mathrm{Re}\ z-k) & i=j\\ -b-a\ \mathrm{Re}\ z & i,j\ \text{nearest neighbours}\\ k & i,j\ \text{next-to-nearest neighbours}\\ 0 & \text{otherwise}\end{cases} \tag{3.3}$$

has the eigenvalues $D(m_1, m_2)$ given by

$$\begin{aligned}D(m_1,m_2)&=6(b+a\ \mathrm{Re}\ z-k)-2(b+a\ \mathrm{Re}\ z)\\&\quad\times\left(\cos\frac{2\pi m_1}{L}+\cos\frac{2\pi m_2}{L}+\cos\frac{2\pi(m_1+m_2)}{L}\right)\\&\quad+2k\left(\cos\frac{2\pi(m_1+2m_2)}{L}+\cos\frac{2\pi(2m_1+m_2)}{L}+\cos\frac{2\pi(m_1-m_2)}{L}\right).\end{aligned} \tag{3.4}$$

The Fourier transform $\boldsymbol{U}_{m_1,m_2}$ of $\boldsymbol{X}_{n_1,n_2}$ is defined as

$$\boldsymbol{U}_{m_1,m_2}=\frac{1}{L}\sum_{n_1,n_2=0}^{L-1}\boldsymbol{X}_{n_1,n_2}\exp\left[-\mathrm{i}\frac{2\pi}{L}(n_1m_1+n_2m_2)\right] \tag{3.5}$$

in terms of which (3.2) becomes

$$\sum_{i,j}D_{ij}\boldsymbol{X}_i\boldsymbol{X}_j=\sum_{m_1,m_2}\sum_{\mu=1}^{d}|U^{\mu}_{m_1,m_2}|^2D(m_1,m_2) \tag{3.6}$$

and all the eigenvalues $D(m_1, m_2)$ are positive if

$$\frac{b+a\ \mathrm{Re}\ z}{k}>\frac{3-\cos 4\pi/L-2\cos 2\pi/L}{2-2\cos\dfrac{2\pi}{L}}\simeq 3-\frac{2\pi^2}{L^2}. \tag{3.7}$$

Then the partition function becomes

$$Z=\frac{\pi^{d(L^2-1)/2}}{2\mathrm{i}\sqrt{\pi}}L^{1+d}\int_{-\mathrm{i}\infty+c}^{\mathrm{i}\infty+c}\mathrm{d}z\exp\left(\frac{z^2L^2}{4}-\frac{d}{2}\sum_{m_1m_2}{}'\ln D(m_1,m_2)\right) \tag{3.8}$$

provided Re z verifies (3.7). The sum Σ' does not include the mode $(m_1, m_2)=0$. The saddle-point equation gives

$$\bar{z}=\frac{da}{L^2k}\sum{}'\left(\frac{b+\bar{z}a}{k}-P(m_1,m_2)\right)^{-1} \tag{3.9}$$

where

$$\begin{aligned}P(m_1,m_2)&=\left[3-\cos\frac{2\pi}{L}(2m_1+m_2)-\cos\frac{2\pi}{L}(m_1+2m_2)-\cos\frac{2\pi}{L}(m_1-m_2)\right]\\&\quad\times\left[3-\cos\frac{2\pi}{L}m_1-\cos\frac{2\pi}{L}m_2-\cos\frac{2\pi}{L}(m_1+m_2)\right]^{-1}.\end{aligned} \tag{3.10}$$

The expression of R^2 can be obtained by introducing in (3.1) a source term as in (2.2). The saddle point approximation of R^2 is

$$R^2 = \frac{d}{L^2} \sum{}' D^{-1}(m_1, m_2)|_{z=\bar{z}}. \tag{3.11}$$

The flat regime will be obtained, as in the polymer case, carrying out the limit of the contour of integration in (3.8) towards the origin of the cut in the z-plane (see (3.7)). Therefore it is quite convenient to isolate in (3.11) the first pole, which defines the origin of the cut:

$$\begin{aligned} R^2 = {} & \frac{dL^2}{L^2 kx} \frac{1}{1-\cos 2\pi/L} + \frac{d}{L^2 k} \sum_{m=2}^{(L-1)/2} \left(1 - \cos\frac{2\pi m}{L}\right)^{-1} \\ & \times \left(\frac{x}{L^2} - P(0,m) + P(0,1)\right)^{-1} \\ & + \frac{2d}{L^2 k} \sum_{m_1, m_2 = 1}^{(L-1)/2} \left(3 - \cos\frac{2\pi m_1}{L} - \cos\frac{2\pi m_2}{L} - \cos\frac{2\pi(m_1+m_2)}{L}\right)^{-1} \\ & \times \left(\frac{x}{L^2} + P(0,1) - P(m_1, m_2)\right)^{-1} \end{aligned} \tag{3.12}$$

with

$$\frac{x}{L^2} = \frac{b + \bar{z}a}{k} - P(0,1). \tag{3.13}$$

The flat regime will correspond to $0 < x \ll 1$, while crumpled configurations occur at different scales of x. The treatment of (3.12) depends on the scale of x considered. *For $x \gg 1$, the expansion of each term of (3.12) does not give positive powers of L (as* it can be directly seen from (3.11)), and to extract terms growing with L from the double sum it is convenient to apply the Euler–MacLaurin summation formula, which gives (see appendix A):

$$R^2 \simeq \frac{d\sqrt{3}}{9\pi(b + \bar{z}a - 3k)} \ln L \qquad x \gg 1 \tag{3.14}$$

In the extreme case $x \gg L^2$, the equations (3.13) and (3.9) respectively become $\bar{z} \gg (3k - b)/a$ and $\bar{z} \simeq daL^2/xk$, so that $k \ll 1$. Since in all regimes

$$l^2 = \frac{\bar{z}}{6a} \tag{3.15}$$

from (3.13)–(3.15) one gets

$$\frac{R^2}{l^2} \sim \frac{2}{\pi\sqrt{3}} \ln L \tag{3.16}$$

which is the extremely crumpled regime. On the other hand, when $x \ll 1$, R^2 can be approximated by

$$R^2 \simeq \frac{dL^2}{2\pi^2 kx} (1 + \mathcal{O}(x)) \tag{3.17}$$

while the Euler–MacLaurin formula is appropriate for studying the saddle-point equation, with the result

$$\bar{z} \simeq \frac{4da}{kx} + \text{constant}\,\frac{a}{k}\ln L \qquad x \ll 1. \tag{3.18}$$

Equation (3.18) and $\bar{z} \simeq (3k-b)/a$ (see (3.13)) establish that $k \gg \sqrt{\ln L}$ if $x \ll 1/\ln L$. Hence

$$\frac{R^2}{l^2} \simeq \frac{3}{4\pi^2} L^2 \qquad k \gg \sqrt{\ln L} \tag{3.19}$$

corresponding to the 'flat regime'. The scale $1 \ll x \ll L^2$ can be shown to correspond to $1 \ll k \ll \sqrt{\ln L}$ with the behaviour of R^2/l^2 intermediate between L^2 and $\ln L$.

The above results imply that, at a fixed value of k, the membrane is stretched on a scale $\xi \sim e^{k^2}$. A suggestion coming from a renormalization group calculation in fluid membrane with bending energy [4] indicates a similar behaviour of ξ. It should be stressed that the calculations presented here are exact and do not assume any scaling to hold *a priori*.

3.2. *The exponent* γ

The entropy-exponent γ is defined by

$$Z \sim \exp(c_1 L^2 + c_2 L + (\gamma - 1)\ln L + \dots) \tag{3.20}$$

and can be calculated in the crumpled phase, which is the only one surviving in the thermodynamic limit. Differently from $c_1, c_2, \dots, \gamma$ is believed to be independent of the connectivity of the network, while in general it depends on the topology and on the shape of the boundary [13]. For example, if a general curved manifold with non-trivial topology is embedded with a Gaussian weight (see [13]), the value found for γ is [14]

$$\gamma - 1 = \frac{d}{6}\chi + \begin{cases} 0 & \text{Dirichlet boundary conditions} \\ -d & \text{Neumann boundary conditions} \end{cases} \tag{3.21}$$

where χ is the Euler characteristic of the manifold. In the case considered here, the use of the periodic boundary conditions should make γ independent of the network connectivity. This is just what happens. In appendix B it is shown that the value $\gamma = 1$, found on the triangular network, remains the same on a square regular bi-dimensional lattice.

3.3. *Some results for a stretched network*

A membrane with cylindric boundary conditions, has been studied in the simple case of a 'sausage' configuration. In this configuration the two bundaries of the network are contracted into two points separated along the y axis by a distance h, which is a measure of the stretching imposed on the membrane.

For simplicity a model on a square regular lattice with L^2 sites is considered. The repulsive interaction is between third neighbours (i.e. second neighbours along the axis) and the action is chosen in such a way to have the family of functions

$$\left\{ \sqrt{\frac{2}{L}} \sin\frac{\pi n y}{L} \sqrt{\frac{1}{L}}\, e^{(i2\pi/L)mx} \right\}_{m=0,L-1;\,n=1,L-1} \tag{3.22}$$

as an orthonormal system of eigenvectors. It is necessary to distinguish the value of R_y^2 and of l_y^2 in the y direction from R_x^2 and l_x^2 in the other direction. The calculations are similar to those previously seen, even if much more algebraically complicated [15], and only some distinctive aspects will be presented. As a consequence of the stretching, in the flat regime, the factor multiplying the L^2 leading term of R_x^2/l_x^2 is less than the corresponding factor calculated with periodic boundary conditions. The parameter h appears in the expression of R_y^2 and l_y^2. Keeping only the leading terms, the results are:

$$R_y^2 \sim A\frac{L^2}{kx} + B\frac{h^2L^4}{x^2} \qquad x \ll 1 \tag{3.23a}$$

$$R_y^2 \sim \frac{1}{\pi}\frac{1}{b+\bar{z}-4k}\log L + h^2\left(\text{constant} + O\left(\frac{1}{L}\right)\right) \qquad x \gg 1 \tag{3.23b}$$

where x is defined as in (3.13) and A, B are positive numbers. The case $x \ll 1$ corresponds to the flat regime, as it will result after the division by $l_y^2 \sim \bar{z}$. The saddle point equation is for $x \ll 1$ $(a=1, b=0)$

$$k = c_1\frac{\ln L}{k} + \frac{h^2}{L} + \frac{20L^2h^2}{x^2} + c_2h^2L^2 - 8\frac{h^2}{x} - h^2c_3 + \frac{c_4}{xk} \tag{3.24}$$

where the c_i are positive constants.

Therefore k must be greater than the largest of $\sqrt{\ln L}$ and h^2L^2. Since $\bar{z} \simeq L^2h^2/x^2$, the leading term of the ratio R_y^2/l_y^2 is still proportional to L^2.

4. The random solid model

The $D=3$ case is studied on a cubic regular lattice with the repulsive interaction between third neighbours (second neighbours along the axis). The partition function can be calculated as in the previous sections. The result is

$$Z = \pi^{(d/2)(L^3-1)}\frac{L^3}{2\mathrm{i}\sqrt{\pi}}\int_{-\mathrm{i}\infty+\mathrm{Re}\,z}^{\mathrm{i}\infty+\mathrm{Re}\,z} \mathrm{d}z\, \mathrm{e}^{L^3 f(z)} \tag{4.1a}$$

$$f(z) = \frac{z^2}{4} - \frac{d}{2L^3}\sum{}' \ln D(m_1, m_2, m_3) \tag{4.1b}$$

$$D(m_1, m_2, m_3) = 6(b+za-k) - 2(b+za)\sum_{\mu=1}^{3}\cos\frac{2\pi}{L}m_\mu + 2k\sum_{\mu=1}^{3}\cos\frac{4\pi}{L}m_\mu \tag{4.1c}$$

$$R^2 \simeq \frac{d}{L^3}\sum{}' D^{-1}(m_1, m_2, m_3)|_{z=\bar{z}}. \tag{4.1d}$$

The saddle-point equation is

$$\bar{z} = \frac{da}{L^3}\sum{}'\left(6 - 2\sum_{\mu=1}^{3}\cos\frac{2\pi}{L}m_\mu\right)\Big/ D(m_1, m_2, m_3)|_{z=\bar{z}} \tag{4.1e}$$

with

$$(b + a\,\mathrm{Re}\,\bar{z})/k > 2(1+\cos 2\pi/L). \tag{4.2}$$

Similarly to the spherical model, in $D=3$ the right-hand side of (4.1e) is finite (for $L\to\infty$) when z approaches its limiting value $(4k-b)/a$ given by (4.2). Therefore there exists a finite k_c given by

$$\frac{4k_c-b}{a}=\frac{da}{k_c}\frac{1}{\pi^3}\prod_{\nu=1}^{3}\int_0^{\pi}\mathrm{d}\theta_\nu\left(4-\frac{3-\Sigma_\mu\cos 2\theta_\mu}{4-\Sigma_\mu\cos\theta_\mu}\right)^{-1} \tag{4.3}$$

that, when $k<k_c$, (4.1e) admits a solution. The corresponding values of R^2 can be easily shown to be constant (crumpled phase). A trick [16] to see what happens when $k>k_c$ is to rewrite (4.1e) as

$$\bar{z}=\frac{6da}{L^3k}\left(\frac{b+\bar{z}a}{k}-P(0,0,1)\right)^{-1}+\frac{da}{L^3k}\Sigma''\left(\frac{b+\bar{z}a}{k}-P(m_1,m_2,m_3)\right)^{-1} \tag{4.4a}$$

where

$$P(m_1,m_2,m_3)=\frac{3-\Sigma_\mu\cos((4\pi/L)m_\mu)}{4-\Sigma_\mu\cos(2\pi/L)m_\mu} \tag{4.4b}$$

and to keep L temporarily finite. If $(b+\bar{z}a)/k-P(0,0,1)\simeq 1/L^3$, then it is negligible in the denominator of (4.4a), the sum Σ'' can be replaced by an integral, and, by using (4.3), the equation (4.4a) becomes

$$\bar{z}=\frac{6da}{L^3k}\left(\frac{b+\bar{z}a}{k}-4\right)^{-1}+\frac{k_c}{k}\frac{4k_c-b}{8a}. \tag{4.5}$$

The only admissible solution (due to inequality (4.2)) of the quadratic equation (4.5) is $\bar{z}=(4k-b)/a$ which says that, for $k>k_c$, $f(z)$ and R^2 are given by (4.1b)-(4.1d) with $z=\bar{z}=(4k-b)/a$. It comes out that $\lim_{L\to\infty}R^2/(l^2L^2)$ is finite for $k>k_c$.

The critical exponent ν describes the behaviour of the mass gap $m^2=(b+\bar{z}a)/k-4$ near the phase transition. If $k>k_c$ $m^2=0$, while if $k\leqslant k_c$

$$m^2\sim(k_c-k)^{2\nu}. \tag{4.6}$$

The equations (4.1e) and (4.3) imply that $m^2\sim(k_c-k)^2$ so that $\nu=1$, as in the spherical model.

5. Conclusions and perspectives

We have presented an exactly solvable model of D-dimensional discrete manifolds randomly embedded in the d-dimensional Euclidean space. The model allows us to calculate explicitly different regimes and to determine the crossover region between the crumpled and flat phases as far as the linear size $N^{1/D}$ of the manifold is large but finite. In particular for $D=2$ this crossover region is located around the value $(\ln N)^{1/2}$ of the 'rigidity constant' k and thus even for macroscopic membranes it could stay at reasonable value of k.

The Hamiltonian (2.1) is a generalization of the Berlin–Kac spin model and it can be seen as the leading contribution in the $1/d$ expansion of the more realistic model

$$H\{\boldsymbol{X}\}=\sum_{\langle ij\rangle}\left\{\frac{\alpha^2}{d}[(\boldsymbol{X}_i-\boldsymbol{X}_j)^2]^2+b(\boldsymbol{X}_i-\boldsymbol{X}_j)^2\right\}-k\sum_{\langle\langle ij\rangle\rangle}(\boldsymbol{X}_i-\boldsymbol{X}_j)^2. \tag{5.1}$$

where now for simplicity the network is a square lattice. In order to show this result, one introduces the auxiliary field φ_{ij} for each couple of nearest-neighbour sites and defines the new Hamiltonian

$$H\{X;\varphi\}=b\sum_{\langle ij\rangle}(X_i-X_j)^2-k\sum_{\langle\langle ij\rangle\rangle}(X_i-X_j)^2+\sum_{\langle ij\rangle}\varphi_{ij}(X_i-X_j)^2\alpha-\frac{d}{4}\sum_{\langle ij\rangle}\varphi_{ij}^2 \tag{5.2}$$

which is such that

$$\prod_{\langle ij\rangle}\int_{-i\infty}^{i\infty}\frac{d\varphi_{ij}}{i\sqrt{4\pi/d}}\,e^{-H\{X;\varphi\}}=e^{-H\{X\}}. \tag{5.3}$$

At variance with the model of the previous sections there is a field variable φ_{ij} for each link instead of a single variable z.

By interchanging the order of integration of φ's and X's, one gets

$$Z=\int\mathscr{D}\varphi\int\mathscr{D}X\,e^{-H(X;\varphi)}\delta(X_c)=\int\mathscr{D}\varphi\,e^{dF(\varphi)} \tag{5.4}$$

where $X_c=\Sigma_1^N{}_iX_i/N$ and in the large d limit one can apply the standard saddle point method. The homogeneous solution $\bar{\varphi}$ of the saddle point equations $\partial F/\partial\varphi_{ij}=0$ is given by

$$\bar{\varphi}=\frac{\alpha}{2L^2}\sum_{m\neq0}\frac{4-2\cos(2\pi m_1/L)-2\cos(2\pi m_2/L)}{D(m;b+\bar{\varphi}\alpha,k)} \tag{5.5}$$

where the D's are the eigenvalues of the quadratic part in the X of (5.2) when $\varphi_{ij}=\bar{\varphi}$.

It is easy to see that (5.5) is exactly analogous (3.9) in the square lattice with $\alpha=2a\sqrt{d}$ and $\bar{\varphi}=\bar{z}/\sqrt{d}$ finite constants in the $d\to\infty$ limit.

Furthermore the free energy and other quantities, such as the mean square radius of gyration, turn out to be the same as the corresponding quantities already calculated for the model (2.1).

Corrections to the leading order in the $1/d$ expansion can be done using methods of the reference [17]. This is what has been done in [18] for a continuous D-dimensional manifold. The most surprising thing is that for $D>2-2/d$ there are the crumpled regime and the flat one surviving in the thermodynamic limit. Corrections of the order $1/d$ in the model (5.1) can be also calculated in a way similar (even if more complicated by the discreteness of the network) to the one developed in the second reference of [18] and will be presented elsewhere.

Acknowledgments

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Appendix A. R^2 in the crumpled regime

The mean squared radius of gyration R^2 can be written as (see (3.11))

$$R^2=\frac{4d}{L^2}\sum_{x=1}^{(L-1)/2}\frac{1}{D(x,0)}+\frac{4d}{L^2}\sum_{x=1}^{(L-1)/2}\sum_{y=1}^{(L-1)/2}\frac{1}{D(x,y)} \tag{A1}$$

which, after repeated application of the Euler–MacLaurin summation formula, becomes

$$\begin{aligned}
R^2=\frac{4d}{L^2}\Bigg[&\int_1^{(L-1)/2}\mathrm{d}x\int_1^{(L-1)/2}\mathrm{d}y\frac{1}{D(x,y)}+\int_1^{(L-1)/2}\mathrm{d}y\frac{1}{D((L-1)/2,y)}\\
&+\int_1^{(L-1)/2}\mathrm{d}y\frac{1}{D(1,y)}+\sum_{l\geqslant 1}\frac{B_{2l}}{2l!}\bigg(2\int_1^{(L-1)/2}\mathrm{d}y\,\partial_x^{(2l-1)}\frac{1}{D(x,y)}\bigg|_{x=(L-1)/2}\\
&-2\int_1^{(L-1)/2}\mathrm{d}y\,\partial_x^{(2l-1)}\frac{1}{D(x,y)}\bigg|_{x=1}\bigg)\\
&+\frac{1}{4}\bigg(\frac{1}{D((L-1)/2,(L-1)/2)}+2\frac{1}{D((L-1)/2,1)}+\frac{1}{D(1,1)}\bigg)\\
&+\sum_{l\geqslant 1}\frac{B_{2l}}{2l!}\bigg(\partial_x^{(2l-1)}\frac{1}{D(x,(L-1)/2)}\bigg|_{x=(L-1)/2}-\partial_x^{(2l-1)}\frac{1}{D(x,1)}\bigg|_{x=1}\\
&+\partial_x^{(2l-1)}\frac{1}{D(x,1)}\bigg|_{x=(L-1)/2}-\partial_x^{(2l-1)}\frac{1}{D(x,(L-1)/2)}\bigg|_{x=1}\bigg)\\
&+\sum_{l\geqslant 1}\sum_{k\geqslant 1}\frac{B_{2l}}{2l!}\frac{B_{2k}}{2k!}\bigg(\partial_x^{(2l-1)}\partial_y^{(2k-1)}\frac{1}{D(x,y)}\bigg|_{x=(L-1)/2,y=(L-1)/2}\\
&-2\partial_x^{(2l-1)}\partial_y^{(2k-1)}\frac{1}{D(x,y)}\bigg|_{x=1,y=(L-1)/2}+\partial_x^{(2l-1)}\partial_y^{(2k-1)}\frac{1}{D(x,y)}\bigg|_{x=1,y=1}\bigg)\\
&+\int_1^{(L-1)/2}\mathrm{d}x\frac{1}{D(0,x)}+\frac{1}{2}\bigg(\frac{1}{D(0,(L-1)/2)}+\frac{1}{D(0,1)}\bigg)\\
&+\sum_{l\geqslant 1}\frac{B_{2l}}{2l!}\bigg(\partial_y^{(2l-1)}\frac{1}{D(0,y)}\bigg|_{y=(L-1)/2}-\partial_y^{(2l-1)}\frac{1}{D(0,y)}\bigg|_{y=1}\bigg)\Bigg]
\end{aligned}\tag{A2}$$

where B_{2l} are the Bernoulli numbers.

All the terms of (A2), except the double integral, remain finite in the $L\to\infty$ limit. In order to show this result, the following approximations of the eigenvalues $D(x,y)$, when $L\gg 1$, will be useful:

$$\begin{aligned}
&D\left(\frac{L-1}{2},\frac{L-1}{2}\right)\sim D\left(\frac{L-1}{2},1\right)\sim D\left(0,\frac{L-1}{2}\right)\sim 8(b+\bar{z}a-k)\\
&D(1,1)\sim\left(\frac{2\pi}{L}\right)^2 6(b+\bar{z}a-3k)\qquad D(0,1)\sim\left(\frac{2\pi}{L}\right)^2 2(b+\bar{z}a-3k).
\end{aligned}\tag{A3}$$

To show that terms such as $L^{-2}\partial^n(1/D(x,y))$ (where $\partial=\partial_x$ or $\partial=\partial_y$) are finite when $L\to\infty$, it may be convenient to write

$$\frac{1}{L^2}\partial^n\frac{1}{D(x,y)}=\frac{1}{L^2}\frac{1}{(D(x,y))^{n+1}}\left(\frac{2\pi}{L}\right)^n G_n(x,y)\tag{A4}$$

where $G_n(x,y)$ is a sum of terms each of them containing n factors chosen between $D\equiv D(x,y)$ and the derivatives of D in such a way that the total number of derivations is n (this property can be easily shown by induction). When $(x,y)=(1,1)$ or $(x,y)=(0,1)$, one sees that $D\sim 1/L^2$ and $\partial D\sim 1/L$. Each term of $G_n(x,y)$ contains powers of $1/L$ at least of order n. Indeed one considers the worst case when $\partial_n D$ with $n>1$

has no powers of $1/L$. A term with derivatives with highest order $n-a$ and with $2x$ factors of ∂D has the structure

$$(\partial^{(n-a)}D)(\partial D)^{2x}(\partial^2 D)^{(a/2)-x}D^{n-1-x-a/2} \tag{A5}$$

when the number of insertions of D is minimum. The resulting total power of $1/L$ is $2n-2-a$ which is equal to or greater than n for $a \leqslant n-2$, independently of x. Therefore the only case to be still considered is $a=n-1$ corresponding to a term proportional to $(\partial D)^n \sim 1/L^n$, which completes the proof.

Terms such as $L^{-2}\int_1^{(L-1)/2} \mathrm{d}x\, \partial_y^n(1/D(x,y))$ can be analysed by also using (A4). Since contributions growing with L could arise only from the lower integration limit when $y=0$ or $y=1$ (see (A3)), the previously examined structure of the function $G_n(x,y)$ and the approximated expressions of $D(x,1)$, $D(x,0)$ for $x \ll 1$ can be used to show that

$$\frac{1}{L^2}\int_1^{(L-1)/2} \mathrm{d}x\, \partial_y^n \frac{1}{D(x,y)} \approx \text{constant} + \mathcal{O}\left(\frac{1}{L}\right). \tag{A6}$$

In similar ways all the other terms of (A2) without derivatives can be shown to be finite when $L\to\infty$ except the double integral. In the new variables $p_1=(2\pi/L)x$, $p_2=(2\pi/L)(x/2+y)2/\sqrt{3}$ the double integral reads

$$\begin{aligned}\frac{4d}{L^2}&\int_1^{(L-1)/2}\int_1^{(L-1)/2} \mathrm{d}x\,\mathrm{d}y \frac{1}{D(x,y)}\\ &=\frac{\sqrt{3}d}{2\pi^2}\int_{2\pi/L}^{\pi(L-1)/L} \mathrm{d}p_1 \int_{p_1/\sqrt{3}+4\pi/\sqrt{3}L}^{p_1/\sqrt{3}+2\pi(L-1)/L\sqrt{3}} \mathrm{d}p_2 \frac{1}{D(p_1,p_2)}\end{aligned} \tag{A7}$$

where

$$\begin{aligned}D(p_1,p_2)&=6(b+\bar{z}a-k)-2(b+\bar{z}a)\left(\cos p_1+2\cos\frac{\sqrt{3}}{2}p_2\cos\frac{p_1}{2}\right)\\ &\quad+2k\left(\cos\sqrt{3}p_2+2\cos\frac{3}{2}p_1\cos\frac{\sqrt{3}}{2}p_2\right)\approx\frac{3}{2}p^2(b+\bar{z}a-3k).\end{aligned} \tag{A8}$$

Then

$$\begin{aligned}\frac{4d}{L^2}&\int_1^{(L-1)/2}\int_1^{(L-1)/2} \mathrm{d}x\,\mathrm{d}y \frac{1}{D(x,y)}\\ &\approx \text{constant}+\frac{d\sqrt{3}}{2\pi^2}\int_{\pi/6}^{\pi/2}\mathrm{d}\theta\int_{1/L} p\,\mathrm{d}p\frac{2}{3p^2(b+\bar{z}a-3k)}\\ &\approx\frac{d\sqrt{3}}{9\pi}\frac{1}{(b+\bar{z}a-3k)}\ln L+\text{constant}+\mathcal{O}\left(\frac{1}{L}\right)\end{aligned} \tag{A9}$$

which is the behaviour of R^2 in the crumpled regime (see (3.14)).

Appendix B. The γ exponent on the triangular and on the square lattices in the crumpled regime

The partition function can be evaluated in the saddle-point approximation. Formula

(3.8) gives

$$Z=\pi^{d/2(L^2-1)}\frac{L^{1+d}}{2\mathrm{i}\sqrt{\pi}}\int_{\mathrm{Re}\,z-\mathrm{i}\infty}^{\mathrm{Re}\,z+\mathrm{i}\infty}\mathrm{d}z\exp\left(\frac{L^2}{2}f(z)\right)$$
$$\approx\pi^{d/2(L^2-1)}\frac{L^{1+d}}{2\sqrt{\pi}}\exp\left(\frac{L^2}{2}f(\bar z)\right)\int_{-\infty}^{\infty}\mathrm{d}t\exp\left(-\frac{L^2}{4}t^2\frac{\mathrm{d}^2f(z)}{\mathrm{d}z^2}\bigg|_{z=\bar z}\right)$$
$$=\pi^{d/2(L^2-1)}\left(\frac{\mathrm{d}^2f(z)}{\mathrm{d}z^2}\bigg|_{z=\bar z}\right)^{-1/2}L^d\exp\left(\frac{L^2}{2}f(\bar z)\right)\qquad\text{(B1)}$$

where $\bar z$ is the real solution of the saddle-point equation (3.9) and

$$f(z)=\frac{z^2}{2}-\frac{d}{L^2}\sum{}'\ln D(m_1,m_2).\qquad\text{(B2)}$$

The term proportional to $\ln L$ in $\Sigma'\ln D(m_1,m_2)$ contributes to the value of γ. The Euler-MacLaurin formula (A2) with $\ln D(x,y)$ instead of $1/D(x,y)$ can be used to evaluate this contribution.

All terms like $\partial^n\ln D(x,y)$ and $\int_1^{(L-1)/2}\mathrm{d}x[\partial_y^{(2n-1)}\ln D(x,y)]$ can be shown to be finite in the $L\to\infty$ limit by the same arguments used for the corresponding terms in (A2). Indeed, for each term (n fixed), $1/D(x,y)$ results in being to the power n instead of $n+1$ as in (A4), and the effect of this change is compensated by the disappearance of the overall factor $1/L^2$ in front of the sum.

The terms $\ln D((L-1)/2,\ (L-1)/2)$, $\ln D(0,(L-1)/2)$, $\ln D(1,(L-1)/2)$ are constant while

$$\ln D(1,1)\approx\ln D(0,1)\approx-2\ln L.\qquad\text{(B3)}$$

Moreover

$$\int_1^{(L-1)/2}\mathrm{d}x\ln D(x,1)\approx\int_1\mathrm{d}x\ln\left[2(b+\bar za-3k)\left(\frac{2\pi}{L}\right)^2(x^2+x+1)\right]$$
$$\approx L\int_{1/L}\mathrm{d}x\ln\left[\left(x+\frac{1}{2L}\right)^2+\frac{3}{4L^2}\right]\approx3\ln L+\text{less divergent terms}\qquad\text{(B4)}$$

$$\int_1^{(L-1)/2}\mathrm{d}x\ln D(x,0)\approx L\int_{1/L}\mathrm{d}x\ln x^2\approx2\ln L+\text{less divergent terms}\qquad\text{(B5)}$$

It remains to calculate the double integral:

$$\int_1^{(L-1)/2}\int_1^{(L-1)/2}\mathrm{d}x\,\mathrm{d}y\ln D(x,y)$$
$$=\frac{L^2\sqrt3}{8\pi^2}\int_{2\pi/L}^{\pi(L-1)/L}\mathrm{d}p_1\int_{(p_1/\sqrt3)+(4\pi/\sqrt3L)}^{(p_1/\sqrt3)+(2\pi/\sqrt3)[(L-1)/L]}\mathrm{d}p_2\ln D(p_1,p_2)$$
$$\approx\frac{L^2\sqrt3}{8\pi^2}\int_{2\pi/L}^{\pi(L-1)/L}\mathrm{d}p_1\int_{(p_1/\sqrt3)+(4\pi/\sqrt3L)}^{(p_1/\sqrt3)+(2\pi/\sqrt3)[(L-1)/L]}$$
$$\times\mathrm{d}p_2\ln\left(\tfrac32(b+\bar za+3k)(p_1^2+p_2^2)\right).\qquad\text{(B6)}$$

In order to extract the term proportional to $\ln L/L^2$ it is convenient to differentiate with respect to L the integrals resulting from (B6) after the integration over p_2. The result is

$$\int_1^{(L-1)/2}\int_1^{(L-1)/2} \mathrm{d}x\,\mathrm{d}y \ln D(x,y) \sim -3\ln L + \text{less divergent terms.} \tag{B7}$$

Inserting (B3)-(B7) into (B2) gives

$$\sum{}' \ln D(m_1, m_2) \sim 2\ln L.$$

Then, in the crumpled regime, the value of γ, as defined in (3.16), is

$$\gamma = 1. \tag{B8}$$

It is noteworthy that the same value of γ is obtained on a square regular lattice. The calculation proceeds as before starting from (B1) and applying the Euler-MacLaurin formula, where now

$$D(m_1, m_2) = 4(b+\bar{z}a-k) - 2(b+\bar{z}a)\left(\cos\frac{2\pi}{L}m_1 + \cos\frac{2\pi}{L}m_2\right) + 2k\left(\cos\frac{4\pi}{L}m_1 + \cos\frac{4\pi}{L}m_2\right) \tag{B9}$$

with $(b+\bar{z}a)/k > 2(1+\cos 2\pi/L)$.

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